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A relation between moduli space of D-branes on orbifolds and Ising model

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Abstract: We study D-branes transverse to an abelian orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. The moduli space of the gauge theory on the D-branes is analyzed by combinatorial calculation based on toric geometry. It is shown that the calculation is related to a problem to count the number of ground states of an antiferromagnetic Ising model. The lattice on which the Ising model is defined is a triangular one defined on the McKay quiver of the orbifold.

Keywords: Lattice Integrable Models, Differential and Algebraic Geometry, D-branes.

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1. Introduction

The moduli space of the gauge theory on D-branes transverse to an orbifold $\mathbb{C}^3/\Gamma$ has been investigated based on the procedure proposed by Douglas, Greene and Morrison [1 – 5]. In this approach the geometry of the moduli space, which is defined by F-flatness and D-flatness conditions, is analyzed in the language of toric variety and the result is described by a cone in a integer lattice $\mathbb{Z}^c$. The dimension c is determined by a calculation of a dual cone which is necessary to make the F-flatness and D-flatness conditions on the same footing. To find the dual cone we must carry out a combinatorial algorithm based on the linear programming, but it is difficult to obtain an explicit result in general. By inspecting closely the calculation of the dual cone, however, one can see that it does not fully use the structure of the gauge theory.

One of the notable feature of the gauge theory is that it is a quiver gauge theory [6, 7]. Information on the gauge theory such as F-flatness and D-flatness conditions are encoded in the McKay quiver of the group Γ [8, 9]. So if we fully make use of the information as a quiver gauge theory, the calculation of the dual cone will become easier. Based on this observation we show that the combinatorial calculation of the dual cone is equivalent to a problem of counting the number of ground states of a certain Ising model on a lattice defined on the McKay quiver of the orbifold. In this paper we consider the orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$, in which case the Ising model is an antiferromagnetic one on a triangular lattice with certain boundary conditions.

In section 2 we review the gauge theory on one D3-brane transverse to a quotient singularity $\mathbb{C}^3/\Gamma$ with Γ an abelian subgroup of SU(3). We also recapitulate the procedure [1] for the analysis of the moduli space $\mathcal{M}$ of the gauge theory. In section 3, we investigate

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the orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$. We show that the calculation of the dual cone can be replaced by a calculation of ground states of an antiferromagnetic Ising model on a triangular lattice constructed from the McKay quiver. In section 4, we present a systematic way to count the number of the ground states by using transfer matrices of the Ising model. We also generalize the analysis to the orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$.

This paper is based on a talk "A relation between quiver variety and Ising model" at the autumn meeting of the Physical Society of Japan, September 2003.

2. The moduli space for one D-brane transverse to an abelian quotient singularity $\mathbb{C}^3/\Gamma$

In this section, we briefly review the gauge theory on one D3-brane transverse to a quotient singularity $\mathbb{C}^3/\Gamma$ with Γ an abelian subgroup of SU(3).

2.1 The gauge theory on the D-brane and the McKay quiver

The gauge theory on one D3-brane transverse to a quotient singularity $\mathbb{C}^3/\Gamma$ is obtained by considering $|\Gamma|$ D3-branes on $\mathbb{C}^3$, and then projecting to $\mathbb{C}^3/\Gamma$. The worldvolume theory on $|\Gamma|$ D3-branes on $\mathbb{C}^3$ is a theory with $\mathrm{U}(|\Gamma|)$ gauge symmetry, containing a vector multiplet A_a $(a = 0, \ldots, 3)$ and three chiral multiplets X^μ $(\mu = 1, 2, 3)$ in the adjoint representation of the gauge group. The components of the gauge field A_a that survive the projection are those which satisfy

$$\gamma(g) A_a \gamma(g)^{-1} = A_a \,, \tag{2.1}$$

where $g \in \Gamma$, and $\gamma(g)$ is the regular representation of Γ. For an abelian group Γ, the regular representations takes the form

$$\gamma(g) = \oplus_{v=0}^{|\Gamma|-1} R_v(g) \,, \tag{2.2}$$

where R_v are one-dimensional irreducible representations of Γ. The condition (2.1) then implies that all elements of A_α must be zero except for diagonal components and hence the gauge group is isomorphic with $\mathrm{U}(1)^{|\Gamma|}$. Note that the overall U(1) acts trivially on all fields, so the effective gauge symmetry is $\mathrm{U}(1)^{|\Gamma|-1}$. The surviving components of X^μ are those which satisfy

$$\gamma(g) X^\mu \gamma(g)^{-1} = R(g)^\mu_\nu X^\nu \,, \tag{2.3}$$

where $R(g)$ is the three-dimensional representation defining the action of Γ on $\mathbb{C}^3$. We choose the three-dimensional representation of the form

$$R(g) = R_{v_1}(g) \oplus R_{v_2}(g) \oplus R_{v_3}(g) \tag{2.4}$$

with $R_{v_1} \otimes R_{v_2} \otimes R_{v_3}$ to be the trivial representation R_0. All elements of X^μ must be zero except for the $3|\Gamma|$ components $X^1_{v\;v+v_1}, X^2_{v\;v+v_2}, X^3_{v\;v+v_3}$, which we denote as

$$x_v = X^1_{v\;v+v_1}\,, \qquad y_v = X^2_{v\;v+v_2}\,, \qquad z_v = X^3_{v\;v+v_3}\,. \tag{2.5}$$

Charges of the v-th U(1) gauge symmetry of x_w, y_w, z_w are given by

$$\begin{aligned} q_v(x_w) &= \delta_{w,v+v_1} - \delta_{w,v}\,, \\ q_v(y_w) &= \delta_{w,v+v_2} - \delta_{w,v}\,, \\ q_v(z_w) &= \delta_{w,v+v_3} - \delta_{w,v} \end{aligned} \tag{2.6}$$

respectively.

Gauge group and matter content of the world volume theory on the D-brane are specified by a McKay quiver. The McKay quiver is a graph with a set of nodes V and a set of arrows A connecting nodes. Each node, which has one to one correspondence with the one-dimensional irreducible representation R_v of Γ, gives rise to a factor U(1) of the gauge group. The set of arrows of the quiver is determined by the following irreducible decomposition

$$R \otimes R_v = \oplus n_{vw} R_w\,. \tag{2.7}$$

Non-negative integer n_{vw} is the number of arrows from node v to node w. For the representation $R(g)$ in (2.4), the decomposition takes the form

$$R \otimes R_v = R_{v+v_1}(g) \oplus R_{v+v_2}(g) \oplus R_{v+v_3}(g)\,, \tag{2.8}$$

thus we have three arrows (a_v, b_v, c_v) leaving the node R_v:

$$\begin{aligned} a_v &: R_v \to R_{v+v_1}\,, \\ b_v &: R_v \to R_{v+v_2}\,, \\ c_v &: R_v \to R_{v+v_3}\,. \end{aligned} \tag{2.9}$$

Note that arrows a_v, b_v, c_v correspond to the fields x_v, y_v, z_v, respectively.

2.2 The moduli space of the gauge theory

The classical moduli space $\mathcal{M}$ of the gauge theory is obtained by imposing the F-flatness and D-flatness conditions of the theory and further dividing by the gauge group $\mathrm{U}(1)^{|\Gamma|-1}$. The F-flatness conditions come from the superpotential

$$W = \mathrm{Tr}\,[X^1, X^2]X^3 \tag{2.10}$$

whose minimization yields the matrix form of the F-term equations

$$[X^\mu, X^\nu] = 0\,. \tag{2.11}$$

In components, the equation takes the form,

$$x_v y_{v+v_1} = y_v x_{v+v_2}\,, \qquad y_v z_{v+v_2} = z_v y_{v+v_3}\,, \qquad z_v x_{v+v_3} = x_v z_{v+v_1}\,. \tag{2.12}$$

Note that the number of independent F-term equations is $2|\Gamma|-2$, and hence these equations define a $|\Gamma|+2$ dimensional complex subvariety $\mathcal{Z}$. Since $\mathcal{Z}$ has the structure of an algebraic variety defined by a collection of monomial relations, it is an affine toric variety and admits an alternative presentation as a holomorphic quotient. If we change notation temporarily as

$$(u_1, \ldots, u_{3|\Gamma|}) = (x_0, \ldots, x_{|\Gamma|-1}, y_0, \ldots, y_{|\Gamma|-1}, z_0, \ldots, z_{|\Gamma|-1}), \tag{2.13}$$

we can solve the equations and express u_l in terms of $|\Gamma|+2$ parameters v_α ($\alpha = 1, \ldots, |\Gamma|+2$);

$$u_l = \prod_\alpha v_\alpha^{K_{l\alpha}}. \tag{2.14}$$

The rows of the $3|\Gamma| \times (|\Gamma|+2)$ matrix $K = (K_{l\alpha})$ are vectors in the lattice $M = \mathbf{Z}^{|\Gamma|+2}$, and define the edges of a cone σ. To describe $\mathcal{Z}$ as a holomorphic quotient, we define a dual cone $\hat{\sigma}$ as follows;

$$\hat{\sigma} = \left\{ \boldsymbol{n} \in \mathbb{R}^{|\Gamma|+2} \,|\, \boldsymbol{m} \cdot \boldsymbol{n} \geq 0 \quad \text{for} \quad \forall \boldsymbol{m} \in \sigma \right\}. \tag{2.15}$$

We introduce a matrix T such that its columns span the dual cone $\hat{\sigma}$. The dimension of the matrix T is $(|\Gamma|+2) \times c$ where c is the number of edges of $\hat{\sigma}$. Each column of T corresponds to a homogeneous coordinate p_λ ($\lambda = 1, \ldots, c$) of the holomorphic quotient description of $\mathcal{Z}$,

$$\mathcal{Z} = \mathbb{C}^c / (\mathbb{C}^*)^{c-|\Gamma|-2}. \tag{2.16}$$

The action of $(\mathbb{C}^*)^{c-|\Gamma|-2}$ on p_α is determined by a $(c-|\Gamma|-2) \times c$ matrix Q_F defined by $Q_F = (\mathrm{Ker}T)^t$.

Note that the variables v_α are solved in terms of p_λ;

$$v_\alpha = \prod_\lambda p_\lambda^{T_{\alpha\lambda}}. \tag{2.17}$$

Combining these equations with (2.14), we obtain the relationship between u_l and p_λ

$$u_l = \prod_\lambda p_\lambda^{K_{l\alpha} T_{\alpha\lambda}}. \tag{2.18}$$

Due to the definition of the dual cone, the exponent $K_{l\alpha}T_{\alpha\lambda} = (KT)_{l\lambda}$ takes value in $\mathbb{Z}_{\geq 0}$.

To obtain the vacuum moduli space $\mathcal{M}$ of the quiver gauge theory, one must further impose $|\Gamma|-1$ D-flatness conditions that exist in the quiver gauge theory from the beginning. The D-term equation for the v-th U(1) group is given by

$$\sum_{l=1}^{3|\Gamma|} q_v(u_l)|u_l|^2 = \zeta_v, \qquad (v = 0, \ldots, |\Gamma|-1), \tag{2.19}$$

where ζ_v is the Fayet-Iliopoulos parameter satisfying $\sum_v \zeta_v = 0$. In the graph-theoretic language, the $|\Gamma| \times 3|\Gamma|$ matrix d with entries $d_{v,l} = q_v(u_l)$ is the incidence matrix of the quiver, where $d_{v,l}$ is -1 if v is the tail of l, $+1$ if v is the head of l and zero otherwise.

Since the gauge group is $\mathrm{U}(1)^{|\Gamma|-1}$ as noted above, one of the D-flatness conditions in (2.19) is redundant. So if we define a $(|\Gamma|-1) \times 3|\Gamma|$ matrix Δ by deleting the last row of d, the $|\Gamma|-1$ independent D-flatness conditions are written as

$$\sum_{l=1}^{3|\Gamma|} \Delta_{v,l}|u_l|^2 = \zeta_v, \qquad (v = 1, \ldots, |\Gamma|-1). \tag{2.20}$$

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To combine these conditions to the expression of $\mathcal{Z}$ (2.16), we must rewrite these conditions to those with respect to the coordinates p_λ, that is, we must find the $\mathrm{U}(1)^{|\Gamma|-1}$ charges for p_λ. The charge matrix Q_D for p_λ is determined from the charge matrix Δ for u_l and the relation between the coordinates u_l and p_λ (2.18) as

$$Q_D(KT)^t = \Delta\,. \tag{2.21}$$

The total number of D-flatness conditions to be imposed on the auxiliary gauge theory is $(c-|\Gamma|-2)+(|\Gamma|-1) = c-3$, and gauge symmetry is $\mathrm{U}(1)^{c-|\Gamma|-2}\times \mathrm{U}(1)^{|\Gamma|-1} = \mathrm{U}(1)^{c-3}$. Thus the moduli space $\mathcal{M}$ takes the form

$$\mathcal{M} = \{(p_1,\ldots,p_c)\in\mathbb{C}^c|(c-3)\text{ D-flatness conditions}\}/\,\mathrm{U}(1)^{c-3}\,. \tag{2.22}$$

Note that the $|\Gamma|-1$ D-flatness conditions in equation (2.19) have Fayet-Iliopoulos parameters θ_i, while $c-|\Gamma|-2$ D-flatness conditions coming from F-flatness conditions do not have such parameters.

Finally the moduli space can be obtained in the form:

$$\mathcal{M} = (\mathcal{Z}-S_\zeta)/(\mathbb{C}^*)^{|\Gamma|-1} = (\mathbb{C}^c - Z_\zeta)/(\mathbb{C}^*)^{c-3}\,, \tag{2.23}$$

where S_ζ, Z_ζ are certain exceptional sets. The action of $(\mathbb{C}^*)^{c-3}$ on $\mathbb{C}^c$ is summarized in the $(c-3)\times c$ matrix

$$Q_t = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix}, \tag{2.24}$$

and the columns of the $3\times c$ matrix $G_t = \mathrm{Ker}(Q_t)^t$ play the role of toric generators of $\mathcal{M}$.

Before ending this section, we show that the matrix G_t is written as

$$G_t = \Pi K T \tag{2.25}$$

if we introduce the $3\times c$ matrix

$$\Pi = \begin{pmatrix} 1 \ldots 1 & 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & 1 \ldots 1 & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 & 1 \ldots 1 \end{pmatrix}, \tag{2.26}$$

where each row has $|\Gamma|$ one and $2|\Gamma|$ zero. Since the matrix $(G_t)^t$ is defined as a kernel of Q_t, what we have to show is

$$Q_t(\Pi K T)^t = \begin{pmatrix} Q_F(\Pi K T)^t \\ Q_D(\Pi K T)^t \end{pmatrix} = 0\,. \tag{2.27}$$

Firstly, the equation $Q_F(\Pi K T)^t = Q_F T^t K^t \Pi^t = 0$ follows from the equation $Q_F T^t = 0$, which is obtained by the definition of Q_F. Secondly,

$$Q_D(\Pi K T)^t = Q_D(KT)^t\Pi^t = \Delta\Pi^t$$

$$= \begin{pmatrix} \sum_w q_1(x_w) & \sum_w q_1(y_w) & \sum_w q_1(z_w) \\ \vdots & \vdots & \vdots \\ \sum_w q_{|\Gamma|-1}(x_w) & \sum_w q_{|\Gamma|-1}(y_w) & \sum_w q_{|\Gamma|-1}(z_w) \end{pmatrix} = 0. \tag{2.28}$$

since the U(1) charge of x_w, for example, satisfy

$$\sum_{w=0}^{|\Gamma|-1} q_v(x_w) = \sum_{w=0}^{|\Gamma|-1} (\delta_{w,v+v_1} - \delta_{w,v}) = 0\,. \tag{2.29}$$

Thus the matrix G_t takes the form

$$G_t = \Pi K T = \begin{pmatrix} \sum_v \boldsymbol{m}(x_v)\cdot\boldsymbol{n}(p_1) & \ldots & \sum_v \boldsymbol{m}(x_v)\cdot\boldsymbol{n}(p_c) \\ \sum_v \boldsymbol{m}(y_v)\cdot\boldsymbol{n}(p_1) & \ldots & \sum_v \boldsymbol{m}(y_v)\cdot\boldsymbol{n}(p_c) \\ \sum_v \boldsymbol{m}(z_v)\cdot\boldsymbol{n}(p_1) & \ldots & \sum_v \boldsymbol{m}(z_v)\cdot\boldsymbol{n}(p_c) \end{pmatrix} \tag{2.30}$$

and the generators of the toric diagram for $\mathcal{M}$ are written as

$$\hat{\boldsymbol{n}}_\lambda = \begin{pmatrix} \sum_v \boldsymbol{m}(x_v)\cdot\boldsymbol{n}(p_\lambda) \\ \sum_v \boldsymbol{m}(y_v)\cdot\boldsymbol{n}(p_\lambda) \\ \sum_v \boldsymbol{m}(z_v)\cdot\boldsymbol{n}(p_\lambda) \end{pmatrix} \qquad (\lambda = 1,\ldots,c)\,. \tag{2.31}$$

What we would like to stress is that to obtain $\boldsymbol{n}_\lambda$ we only need the inner product between $\boldsymbol{m}(z_v)$ and $\boldsymbol{n}(p_\lambda)$ and the expression $\boldsymbol{n}_\lambda$ itself is not necessary. Based on this observation, we propose a method to calculate the matrix G_t without solving the dual cone. Our strategy is as follows; we start with the following relation between u and p

$$u_l = \prod_{\lambda=1}^{c} p_\lambda^{\alpha_{l\lambda}}\,, \tag{2.32}$$

although the number c is not determined yet. Next we impose the F-flatness condition on the indices $\alpha_{l\lambda} \in \mathbb{Z}_{\geq 0}$ and count all possible combinations of indices. The number c is determined as the number of independent combinations which satisfies the F-flatness conditions.

3. $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$

In this section we consider the gauge theory on the worldvolume of a D3-brane transverse to the quotient singularity $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ [10 − 13]. We show that the calculation of the dual cone is replaced by a tiling problem of a torus by rhombus, and a certain dimer problem. (Similar obserbation was made in [14].) We furthermore show that it is equivalent to a calculation of ground states of an antiferromagnetic Ising model on a triangular lattice defined on the McKay quiver.

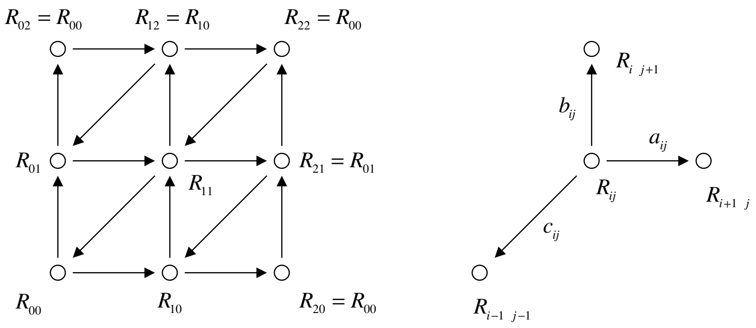

Figure 1: McKay quiver of the quotient singularity $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

3.1 Configuration of indices on the McKay quiver

The group $\mathbb{Z}_n \times \mathbb{Z}_n$ has n^2 one dimensional irreducible representations R_{ij}, and apart from the overall U(1), we have a $\mathrm{U}(1)^{n^2-1}$ gauge symmetry. If we choose the three-dimensional representation as $R = R_{1\,0} \oplus R_{0\,1} \oplus R_{n-1\,n-1}$, the generators g_1, g_2 of the group $\mathbb{Z}_n \times \mathbb{Z}_n$ act on $\mathbb{C}^3$ as

$$\begin{aligned} g_1 &: (X^1, X^2, X^3) \to (\omega X^1, X^2, \omega^{-1} X^3)\,, \\ g_2 &: (X^1, X^2, X^3) \to (X^1, \omega X^2, \omega^{-1} X^3)\,, \end{aligned} \tag{3.1}$$

where $\omega = e^{\frac{2\pi i}{n}}$. Then we have the following decomposition

$$R \otimes R_{ij} = R_{i+1\,j} \oplus R_{i\,j+1} \oplus R_{i-1\,j-1}\,, \tag{3.2}$$

and hence there are three types of arrows

$$\begin{aligned} a_{ij} &: R_{ij} \to R_{i+1\,j}\,, \\ b_{ij} &: R_{ij} \to R_{i\,j+1}\,, \\ c_{ij} &: R_{ij} \to R_{i-1\,j-1}\,. \end{aligned} \tag{3.3}$$

The McKay quiver of the quotient singularity $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ is depicted in figure 1.

This model has $3n^2$ variables x_{ij}, y_{ij}, z_{ij}, which corresponds to the arrows a_{ij}, b_{ij}, c_{ij} respectively. The F-flatness conditions on these variables take the form

$$\begin{aligned} x_{ij}\, y_{i+1\,j} &= y_{ij}\, x_{i\,j+1}\,, \\ y_{ij}\, z_{i\,j+1} &= z_{ij}\, y_{i-1\,j-1}\,, \\ z_{ij}\, x_{i-1\,j-1} &= x_{ij}\, z_{i+1\,j}\,. \end{aligned} \tag{3.4}$$

As explained in the last section, we introduce new variables p_λ $(\lambda = 1, \ldots, c)$ which are related to the original variables x_{ij}, y_{ij}, z_{ij} $(i, j = 0, \ldots, n-1)$ by the following equations,

$$\begin{aligned} x_{ij} &= p_1^{\alpha_{ij}^1}\, p_2^{\alpha_{ij}^2}\, \cdots\, p_c^{\alpha_{ij}^c}\,, \\ y_{ij} &= p_1^{\beta_{ij}^1}\, p_2^{\beta_{ij}^2}\, \cdots\, p_c^{\beta_{ij}^c}\,, \end{aligned}$$

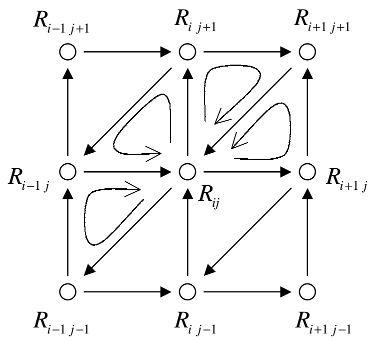

Figure 2: F-flatness conditions imply the equivalence among triangles in the McKay quiver.

$$z_{ij} = p_1^{\gamma_{ij}^1}\, p_2^{\gamma_{ij}^2}\, \cdots\, p_c^{\gamma_{ij}^c}\,, \tag{3.5}$$

where the indices $\alpha_{ij}^\lambda, \beta_{ij}^\lambda, \gamma_{ij}^\lambda \in \mathbb{Z}_{\geq 0}$ are subject to the F-flatness conditions. If we consider the quantity $x_{ij}\, y_{i+1\,j}\, z_{i+1\,j+1}$, we obtain the following relations

$$\begin{aligned} x_{ij}\, y_{i+1\,j}\, z_{i+1\,j+1} &= y_{ij}\, x_{i\,j+1}\, z_{i+1\,j+1} \\ &= y_{ij}\, z_{i\,j+1}\, x_{i-1\,j} \\ &= z_{ij}\, y_{i-1\,j-1}\, x_{i-1\,j}\,, \end{aligned} \tag{3.6}$$

due to the F-flatness conditions (3.4). In terms of the indices $\alpha_{ij}^\lambda, \beta_{ij}^\lambda, \gamma_{ij}^\lambda$, these relations are written as

$$\begin{aligned} \alpha_{ij}^\lambda + \beta_{i+1\,j}^\lambda + \gamma_{i+1\,j+1}^\lambda &= \beta_{ij}^\lambda + \alpha_{i\,j+1}^\lambda + \gamma_{i+1\,j+1}^\lambda \\ &= \beta_{ij}^\lambda + \gamma_{i\,j+1}^\lambda + \alpha_{i-1\,j}^\lambda \\ &= \gamma_{ij}^\lambda + \beta_{i-1\,j-1}^\lambda + \alpha_{i-1\,j}^\lambda\,. \end{aligned} \tag{3.7}$$

To investigate sets of indices $\alpha_{ij}^\lambda, \beta_{ij}^\lambda, \gamma_{ij}^\lambda$ satisfying these conditions, it is useful to consider configurations of indices on the McKay quiver. As noted above, each variables x_{ij}, $y_{i+1\,j}$ and $z_{i+1\,j+1}$ corresponds to arrows a_{ij}, $b_{i+1\,j}$ and $c_{i+1\,j+1}$. Thus in terms of the arrows, the relations (3.6) are written as

$$\begin{aligned} a_{ij}\, b_{i+1\,j}\, c_{i+1\,j+1} &= b_{ij}\, a_{i\,j+1}\, c_{i+1\,j+1} \\ &= b_{ij}\, c_{i\,j+1}\, a_{i-1\,j} \\ &= c_{ij}\, b_{i-1\,j-1}\, a_{i-1\,j}\,. \end{aligned} \tag{3.8}$$

These relations imply that all closed paths encircling triangles of the McKay quiver are equivalent as depicted in figure 2.

If we assign the indices α_{ij}^λ, β_{ij}^λ and γ_{ij}^λ to arrows a_{ij}, $b_{i+1\,j}$ and $c_{i+1\,j+1}$ respectively, we obtain a configuration of indices on the McKay quiver. The conditions (3.7) imply that the sum of indices on every triangle of the McKay quiver must be equal. Thus we only need to consider configurations of indices satisfying this condition. We denote the sum of the indices of each triangle by s (See figure 3).

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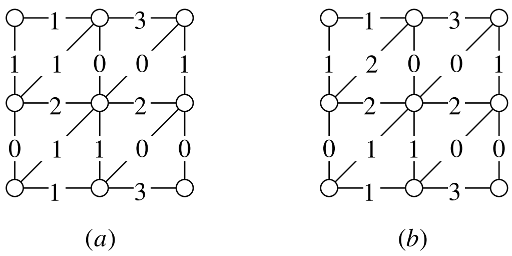

Figure 3: Configurations of indices on the McKay quiver for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. (a) is a configuration with $s = 3$. (b) is not allowed since sums of indices are different from triangle to triangle.

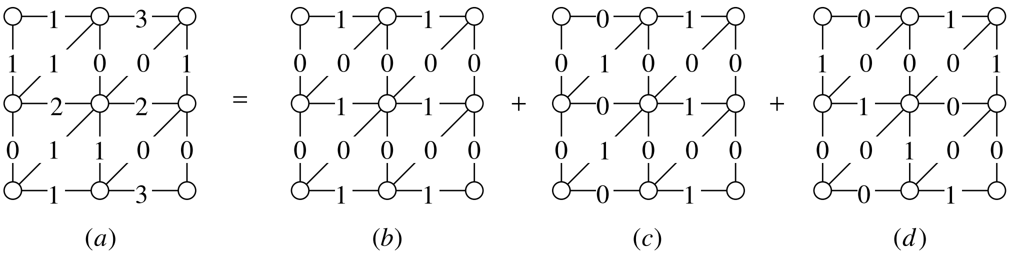

Figure 4: A configuration with $s = 3$ is written as a superposition of three configurations with $s = 1$.

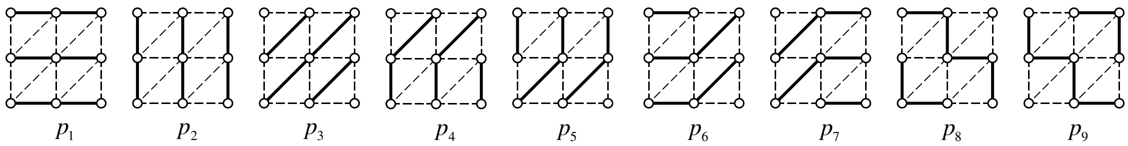

Figure 5: Configurations with $s = 1$ on the McKay quiver for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. Bold lines represent links assigned index one and thin lines represent links assigned index zero. Each configuration corresponds to a coordinate p_λ.

Here an important point is that any configuration with the sum s can be written as a "superposition" of s configurations with $s = 1$ (See figure 4). Based on this fact we assert that each configuration with $s = 1$ corresponds to a coordinate p_λ and hence c is the number ofdifferent configurations with $s = 1$. Thus what we have to do is to enumerate all configurations of indices with $s = 1$.

As an example we depict all configurations of indices with $s = 1$ for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ in figure 5.

There are nine configurations with $s = 1$, which means that $c = 9$. By using the equation (3.5), we obtain the relation between x_{ij}, y_{ij}, z_{ij} and $p_1, \ldots, p_9$ as follows;

$$
\begin{aligned}
x_{00} &= p_1p_6p_8\,, & x_{10} &= p_1p_7p_9\,, & x_{01} &= p_1p_6p_9\,, & x_{11} &= p_1p_7p_8\,,\\
y_{00} &= p_2p_4p_8\,, & y_{10} &= p_2p_4p_9\,, & y_{01} &= p_2p_5p_9\,, & y_{11} &= p_2p_5p_8\,,\\
z_{00} &= p_3p_4p_6\,, & z_{10} &= p_3p_4p_7\,, & z_{01} &= p_3p_5p_6\,, & z_{11} &= p_3p_5p_7\,.
\end{aligned}
\tag{3.9}
$$

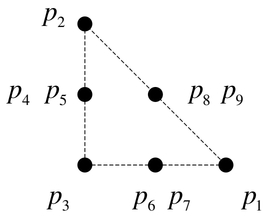

Figure 6: Toric diagram for the model $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. We assign a coordinate p_λ to each toric generator.

From the equation (2.18), we obtain a matrix KT as

$$KT = (\boldsymbol{m}_l \cdot \boldsymbol{n}_\lambda) = \left(\begin{array}{cccccccccc} & \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 & \boldsymbol{n}_4 & \boldsymbol{n}_5 & \boldsymbol{n}_6 & \boldsymbol{n}_7 & \boldsymbol{n}_8 & \boldsymbol{n}_9 \\ \boldsymbol{m}_1\ (x_{00}) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \boldsymbol{m}_2\ (x_{10}) & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \boldsymbol{m}_3\ (x_{01}) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \boldsymbol{m}_4\ (x_{11}) & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \boldsymbol{m}_5\ (y_{00}) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \boldsymbol{m}_6\ (y_{10}) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \boldsymbol{m}_7\ (y_{01}) & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \boldsymbol{m}_8\ (y_{11}) & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \boldsymbol{m}_9\ (z_{00}) & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \boldsymbol{m}_{10}\ (z_{10}) & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \boldsymbol{m}_{11}\ (z_{01}) & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \boldsymbol{m}_{12}\ (z_{11}) & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right). \tag{3.10}$$

This result is the same as that obtained by the calculation of dual cone.

By defining the charge projection matrix Π as

$$\Pi = \begin{pmatrix} 1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1 \end{pmatrix}, \tag{3.11}$$

we obtain nine toric generators $\hat{\boldsymbol{n}}_\lambda$ according to the equation (2.30),

$$G_t = \Pi KT = \left(\begin{array}{ccccccccc} \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{n}}_3 & \hat{\boldsymbol{n}}_4 & \hat{\boldsymbol{n}}_5 & \hat{\boldsymbol{n}}_6 & \hat{\boldsymbol{n}}_7 & \hat{\boldsymbol{n}}_8 & \hat{\boldsymbol{n}}_9 \\ 4 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 4 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 2 & 2 & 2 & 2 & 0 & 0 \end{array} \right). \tag{3.12}$$

Thus we finally obtain the toric diagram as depicted in figure 6.

3.2 Enumeration of configuration of indices on the McKay quiver

Now we would like to present a systematic way to count the configurations of indices with $s = 1$. As we will see below, the configuration of indices is translated to other systems such as a tiling problem, a dimer problem and an Ising model.

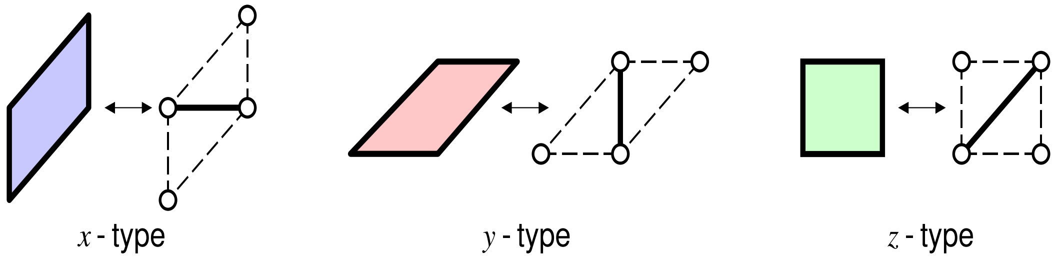

Figure 7: Three types of rhombuses. Each rhombus should be put on the McKay quiver such that its diagonal lies on the indices with one.

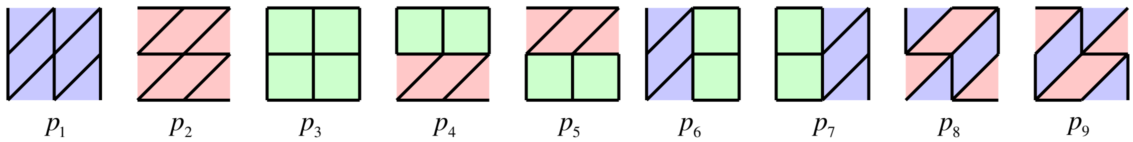

Figure 8: Tiling a torus with size 2×2 by rhombus.

Firstly, we show that enumerating configurations of indices with $s = 1$ on the McKay quiver for $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ is equivalent to a certain tiling problem. What is tiled is a torus with size $n \times n$ and tiles are three types of rhombuses depicted in figure 7.

Each rhombus should be put on the McKay quiver such that its diagonal lies on the link with indices one and its boundary on the link with indices zero. Then the conditions (3.7) on the indices assures that rhombuses cover the torus and do not overlap each other. The fact that configurations of indices with the sum s is a superposition of s configurations with $s = 1$ corresponds to the fact that a s-tiple tiling is decomposed to s tilings with $s = 1$. As an example, we depict all tilings for the model $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ in figure 8.

Secondly, the configuration of indices is translated to a dimer problem on a hexagonal lattice. The vertices of the lattice lie on the center of the triangles of the McKay quiver and dimers connect neighboring vertices such that they intersect to links with indices one.

Thirdly, the problem to enumerate configurations of indices with $s = 1$ is equivalent to a problem to count ground states of an Ising model on a lattice defined on the quiver diagram. Lattice sites of this model are the nodes R_{ij} of the McKay quiver, and spin variables $\pm$ at the nodes interact through neighboring links connected by arrows. Thus the lattice is a triangular one with $n \times n$ sites on a torus. We consider an antiferromagnetic Ising model on this lattice. Although an antiferromagnetic Ising model favors anti-parallel links $(+-)$ and $(-+)$, there must exist parallel links $(++)$ and $(--)$ due to frustration since the lattice is triangular. Therefore every triangle in the lattice has two parallel links and one anti-parallel link in the ground states. By deleting parallel links from the lattice, we obtain a diagram of anti-parallel links. Anti-parallel links form rhombuses because of the property of the ground state of the Ising model. Thus each ground state of the Ising model corresponds to a tiling by rhombuses discussed above.

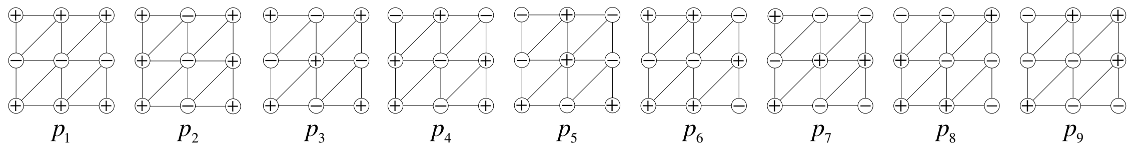

Figure 9: Ground states with $\sigma_{00} = +$ of the antiferromagnetic Ising model on the lattice of the McKay quiver for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

Note that not only periodic boundary condition but also ant-periodic conditions are allowed. Thus there are four boundary conditions

$$\begin{array}{llll} (\mathrm{p},\mathrm{p}) & \sigma_{ij} = \sigma_{i+n\,j}\,, & \sigma_{ij} = \sigma_{i\,j+n}\,, \\ (\mathrm{p},\mathrm{a}) & \sigma_{ij} = \sigma_{i+n\,j}\,, & \sigma_{ij} = -\sigma_{i\,j+n}\,, \\ (\mathrm{a},\mathrm{p}) & \sigma_{ij} = -\sigma_{i+n\,j}\,, & \sigma_{ij} = \sigma_{i\,j+n}\,, \\ (\mathrm{a},\mathrm{a}) & \sigma_{ij} = -\sigma_{i+n\,j}\,, & \sigma_{ij} = -\sigma_{i\,j+n}\,, \end{array} \tag{3.13}$$

where "p" and "a" represent periodic and anti-periodic boundary conditions. An important point is that interchange of + spin and − spin gives the identical configuration, the correspondence between ground states of the Ising model and tilings is two to one.

Ground states of the Ising model on the McKay quiver for $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ is depicted in figure 9. The boundary conditions for $\{p_1, p_2, p_3\}$, $\{p_4, p_5\}$, $\{p_6, p_7\}$ and $\{p_8, p_9\}$ are (p, p), (p, a), (a, p) and (a, a), respectively.

4. Counting the number of the ground states of the Ising model

In this section we count the number of ground states of the Ising model introduced in the last section.

The energy E of the Ising model on the triangular lattice is given by

$$E = J \sum_{i,j=0}^{n-1} (\sigma_{ij}\sigma_{i+1j} + \sigma_{ij}\sigma_{ij+1} + \sigma_{i+1j+1}\sigma_{ij})\,. \tag{4.1}$$

For an antiferromagnetic Ising model, the coupling constant J is positive. The partition function of the Ising model is

$$Z_{n\times n} = \sum_{\{\sigma_{ij}\}} \mathrm{e}^{-\beta E}\,, \tag{4.2}$$

where the sum is taken over all configurations of spins $\{\sigma_{ij}\}$ which satisfy (anti-) periodic boundary conditions in the horizontal and vertical directions. Therefore the partition function is decomposed into a sum of partition functions with the four boundary conditions,

$$Z_{n\times n} = Z^{(\mathrm{pp})}_{n\times n} + Z^{(\mathrm{pa})}_{n\times n} + Z^{(\mathrm{ap})}_{n\times n} + Z^{(\mathrm{aa})}_{n\times n}\,. \tag{4.3}$$

In $Z^{(\mathrm{pp})}_{n\times n}$, for example, summation is restricted to configurations of spins $\{\sigma_{ij}\}$ with the (p, p) boundary condition.

To calculate $Z^{(\mathrm{pp})}_{n\times n}$, we introduce a transfer matrix $T^{(\mathrm{p})}_n$ as follows;

$$
\begin{aligned}
Z^{(\mathrm{pp})}_{n\times n} &= \sum_{\{\sigma_{ij}\}} \prod_{j=0}^{n-1}\prod_{i=0}^{n-1} \mathrm{e}^{-\beta J\left[\frac{1}{2}(\sigma_{ij}\sigma_{i+1j}+\sigma_{ij+1}\sigma_{i+1j+1})+\sigma_{ij}\sigma_{ij+1}+\sigma_{i+1j+1}\sigma_{ij}\right]} \\
&= \sum_{\{\sigma_{ij}\}} \prod_{j=0}^{n-1} (T^{(\mathrm{p})}_n)_{\sigma_j\sigma_{j+1}} \\
&= \mathrm{Tr}\,(T^{(\mathrm{p})}_n)^n\,. \qquad (4.4)
\end{aligned}
$$

The suffix σ_j of the transfer matrix $T^{(\mathrm{p})}_n$ represents a set of spin variables of the j-th row $\{\sigma_{0j},\ldots,\sigma_{n-1j}\}$, so $T^{(\mathrm{p})}_n$ has $2^n\times 2^n$ elements. We use the lexicographic order for the suffix of the matrix. Using the transfer matrix $T^{(\mathrm{p})}_n$, the partition function for the (pa) boundary condition is written as

$$Z^{(\mathrm{pa})}_{n\times n} = \mathrm{Tr}\,[(T^{(\mathrm{p})}_n)^n R_n]\,, \qquad (4.5)$$

where the $2^n\times 2^n$ matrix

$$R_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes n} \qquad (4.6)$$

is inserted in order to realize the anti-periodic boundary conditions in the vertical direction:

$$\sigma_{jn} = -\sigma_{j0}\,. \qquad (4.7)$$

Similarly, the partition functions for the boundary conditions (ap) and (aa) are written as

$$
\begin{aligned}
Z^{(\mathrm{ap})}_{n\times n} &= \mathrm{Tr}(T^{(\mathrm{a})}_n)^n\,, \qquad (4.8)\\
Z^{(\mathrm{aa})}_{n\times n} &= \mathrm{Tr}[(T^{(\mathrm{a})}_n)^n R_n]\,, \qquad (4.9)
\end{aligned}
$$

where $T^{(\mathrm{a})}_n$ is the transfer matrix for the anti-periodic boundary condition in the horizontal direction:

$$\sigma_{nj} = -\sigma_{0j}\,. \qquad (4.10)$$

What we have to do next is to find the transfer matrices $T^{(\mathrm{p})}_n$ and $T^{(\mathrm{a})}_n$. For this purpose, we consider configurations of spins on the $2\times n$ lattice depicted in figure 10(a) with free boundary conditions.

The $2^n\times 2^n$ transfer matrix T_n corresponding to this configuration is written as

$$(T_n)_{\sigma_j\sigma_{j+1}} = \prod_{i=0}^{n-2} \mathrm{e}^{-\beta J\left[\frac{1}{2}(\sigma_{ij}\sigma_{i+1j}+\sigma_{ij+1}\sigma_{i+1j+1})+\sigma_{ij}\sigma_{ij+1}+\sigma_{i+1j+1}\sigma_{ij}\right]}. \qquad (4.11)$$

This matrix satisfies a following recursion relation

$$(T_n)_{\sigma\sigma'} = (T_{n-1}\otimes H_1)_{\sigma\sigma'}(H_{n-2}\otimes T_2)_{\sigma\sigma'}\,, \qquad (4.12)$$

where the $2^k\times 2^k$ matrix H_k is given by

$$H_k = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{\otimes k}. \qquad (4.13)$$

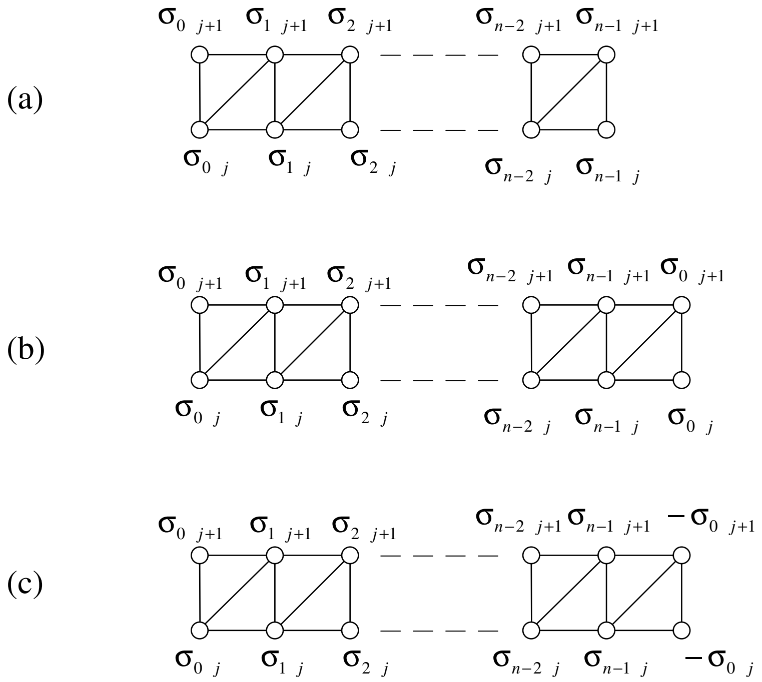

Figure 10: Lattice.

By using the explicit form of the matrix T_2:

$$T_2 = \begin{pmatrix} e^{-3K} & 1 & 1 & e^{K} \\ e^{-2K} & e^{K} & e^{K} & e^{2K} \\ e^{2K} & e^{K} & e^{K} & e^{-2K} \\ e^{K} & 1 & 1 & e^{-3K} \end{pmatrix} \tag{4.14}$$

with $K = \beta J$, we are able to obtain the matrix T_n.

Configurations of spins with the periodic boundary condition in the horizontal direction are realized by adding a pair of spins σ_{00} and σ_{01} on the right side of the lattice as depicted in figure 10(b). Thus the transfer matrix $T_n^{(\mathrm{p})}$ is obtained from T_n by adding the contribution from the last plaquet in figure 10(b). Therefore we obtain

$$(T_n^{(\mathrm{p})})_{\sigma_j \sigma_{j+1}} = (T_n)_{\sigma_j \sigma_{j+1}} (P_n)_{\sigma_j \sigma_{j+1}} \tag{4.15}$$

with

$$(P_n)_{\sigma_j \sigma_{j+1}} = e^{-\beta J \left[\frac{1}{2}(\sigma_{n-2\,j}\sigma_{0\,j} + \sigma_{n-2\,j+1}\sigma_{0\,j+1}) + \sigma_{n-2\,j}\sigma_{n-2\,j+1} + \sigma_{0\,j+1}\sigma_{n-2\,j}\right]}. \tag{4.16}$$

By using the matrix T_2, the $2^n \times 2^n$ matrix P_n is written as

$$P_n = X_n^t (T_2 \otimes H_{n-2}) X_n \tag{4.17}$$

with

$$X_n = (X_2 \otimes \mathbf{1}_2^{n-2})(\mathbf{1}_2 \otimes X_{n-1}), \qquad X_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.18}$$

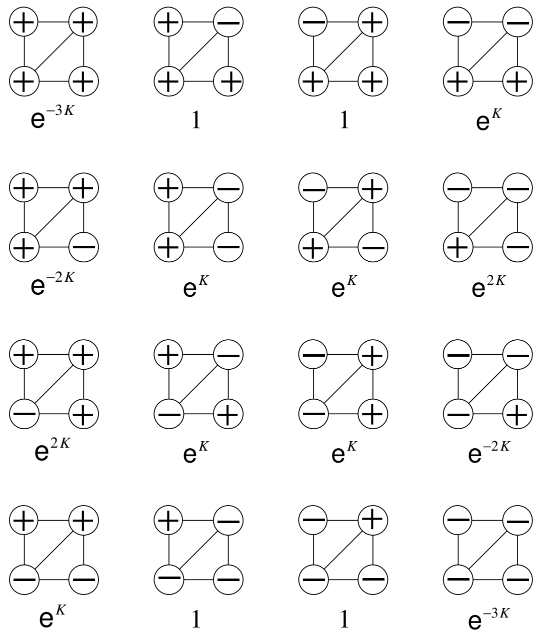

Figure 11: The matrix elements of T_2.

For example, the matrix $T_2^{(\mathrm{p})}$ is given by

$$T_2^{(\mathrm{p})} = \begin{pmatrix} \mathrm{e}^{-6k} & 1 & 1 & \mathrm{e}^{2K} \\ 1 & \mathrm{e}^{2K} & \mathrm{e}^{2K} & 1 \\ 1 & \mathrm{e}^{2K} & \mathrm{e}^{2K} & 1 \\ \mathrm{e}^{2k} & 1 & 1 & \mathrm{e}^{-6K} \end{pmatrix}. \tag{4.19}$$

The transfer matrix $T_n^{(\mathrm{a})}$ for the antiperiodic boundary condition along the horizontal direction is obtained from T_n by adding the contribution from the last plaquet in figure 10(c). Thus $T_n^{(\mathrm{a})}$ is written as

$$(T_n^{(\mathrm{a})})_{\sigma_j \sigma_{j+1}} = (T_n)_{\sigma_j \sigma_{j+1}} (A_n)_{\sigma_j \sigma_{j+1}} \tag{4.20}$$

with

$$(A_n)_{\sigma_j \sigma_{j+1}} = \mathrm{e}^{-\beta J \left[\frac{1}{2} (-\sigma_{n-2\,j}\sigma_{0\,j} - \sigma_{n-2\,j+1}\sigma_{0\,j+1}) + \sigma_{n-2\,j}\sigma_{n-2\,j+1} - \sigma_{0\,j+1}\sigma_{n-2\,j} \right]}. \tag{4.21}$$

The $2^n \times 2^n$ matrix A_n is given by

$$A_n = Z_n^t P_n Z_n\,, \tag{4.22}$$

where

$$Z_n = \begin{pmatrix} 0 & 1_{2^{n-1}} \\ 1_{2^{n-1}} & 0 \end{pmatrix} \tag{4.23}$$

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makes the boundary condition antiperiodic in the vertical direction. For example, the matrix $T_2^{(\mathrm{a})}$ is given by

$$T_2^{(\mathrm{a})} = \begin{pmatrix} \mathrm{e}^{-2k} & \mathrm{e}^{2K} & \mathrm{e}^{-2K} & \mathrm{e}^{2K} \\ \mathrm{e}^{-2K} & \mathrm{e}^{-2K} & \mathrm{e}^{2K} & \mathrm{e}^{2K} \\ \mathrm{e}^{2K} & \mathrm{e}^{2K} & \mathrm{e}^{-2K} & \mathrm{e}^{-2K} \\ \mathrm{e}^{2k} & \mathrm{e}^{-2K} & \mathrm{e}^{2K} & \mathrm{e}^{-2K} \end{pmatrix}. \tag{4.24}$$

Combining these results, the partition function is written as

$$Z_{n\times n} = \mathrm{Tr}[\{(T_n^{(\mathrm{p})})^n + (T_n^{(\mathrm{a})})^n\}(1+R_n)]\,. \tag{4.25}$$

For example, we obtain

$$Z_{2\times 2} = 18\mathrm{e}^{4K} + 32 + 12\mathrm{e}^{-4K} + 2\mathrm{e}^{-12K}\,. \tag{4.26}$$

The first term proportional to e^{4K} corresponds to the ground states. The coefficient 18 means that the number of ground states of the Ising model is 18. Due to the two to one correspondence noted in the last section, the number c of the toric generators is 9.

So far we have calculated the number c of the toric generators by computing the partition function of the Ising model. In the language of the tiling problem, c is the number of tilings of the torus by three types of rhombi. In fact, however, this is not the only information we can extract from the calcuration of the partition function; we are able to count the numbers of x- y- and z- type rhombuses contained in each tiling. For this purpose, we consider which rhombus corresponds to each plaquet in figure 11. For example, the top left plaquet (1-1 element) in figure 11 does not corresponds to any combination of rhombuses since the three spins of each triangle have the same spins, while the plaquet right to this one (1-2 element) corresponds to a combination of the upper half of the x-type rhombus and the right half of the y-type rhombus. In this way, we obtain combinations of rhombuses as depicted in figure 12.

Now we assign weights that count the number of rhombuses to each element in figure 12. For example, we assign zero to the top left element and $\sqrt{xy}$ to the right to this one. These data are summarized in in a matrix as:

$$\hat{T}_2 = \begin{pmatrix} 0 & \sqrt{xy} & 0 & \sqrt{xx} \\ 0 & \sqrt{yy} & z & \sqrt{xy} \\ \sqrt{xy} & z & \sqrt{yy} & 0 \\ \sqrt{xx} & 0 & \sqrt{xy} & 0 \end{pmatrix}. \tag{4.27}$$

Now we introduce a function $\hat{Z}_{n\times n}$ obtained by replacing T_2 in $Z_{n\times n}$ with $\hat{T}_2$. Then, for example, we obtain

$$\frac{1}{2}\hat{Z}_{2\times 2} = x^4 + y^4 + z^4 + 2y^2z^2 + 2z^2x^2 + 2x^2y^2 + \cdots\,, \tag{4.28}$$

where we have omitted terms which do not correspond to the ground states. The first term x^4, for example, implies that there is a tiling with four x-type rhombuses, and the last term $2x^2y^2$ implies that there are two tilings with two x-type rhombuses and two y-type

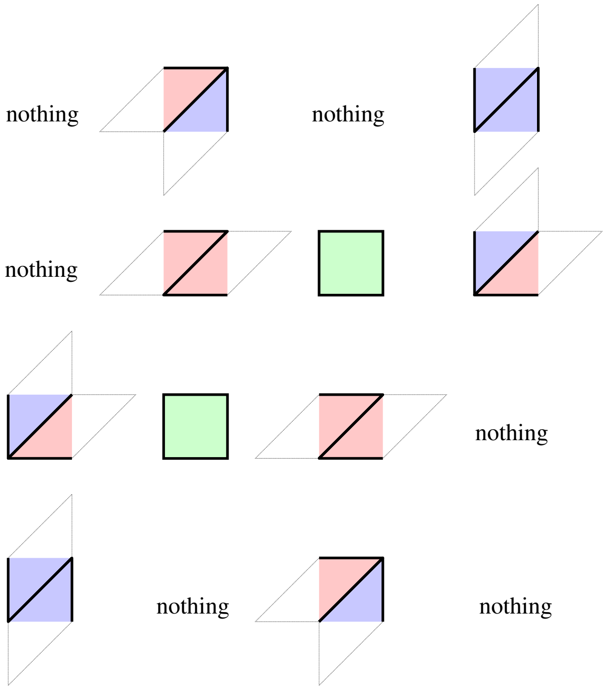

Figure 12: Combinations of rhombuses corresponding to each plaquet in figure 11.

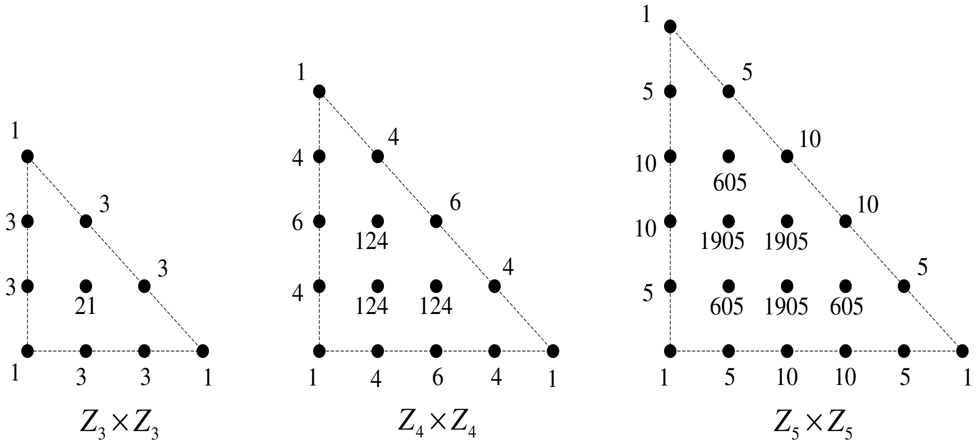

Figure 13: The numbers of toric generators on each vertex.

rhombuses. Note that the numbers of x- y- and z- type rhombuses correspond to the three entries of the toric generators $\hat{\boldsymbol{n}}_\lambda$ in the equation (2.30), and hence we are able to draw toric diagrams. In figure 13, the numbers of toric generators on each vertex are depicted for $n = 3, 4, 5$.

So far we have considered only the models of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$, but it is straightforward to generalize the analysis to the models of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$ The partition function for the model $Z_{n\times m}$ is given by

$$Z_{n\times m} = Z^{(\mathrm{pp})}_{n\times m} + Z^{(\mathrm{pa})}_{n\times m} + Z^{(\mathrm{ap})}_{n\times m} + Z^{(\mathrm{aa})}_{n\times m}\,, \tag{4.29}$$

	1	2	3	4	5	6
1	3	5	9	17	33	65
2	5	9	17	33	65	129
3	9	17	42	113	309	860
4	17	33	113	417	1537	5793
5	33	65	309	1537	7623	38405
6	65	129	860	5793	38405	263640
7	129	257	2445	22081	193849	1849073
8	257	513	7073	84609	984577	13034049
9	513	1025	20706	326273	5022543	92168924
10	1025	2049	61097	1265793	25659185	657904329
11	2049	4097	181245	4933121	131347909	4735839173

Table 1: The number c of homogeneous coordinates for the model $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$.

where

$$Z^{(\mathrm{pp})}_{n\times m} = \mathrm{Tr}(T^{(\mathrm{p})}_n)^m \,, \tag{4.30}$$
$$Z^{(\mathrm{pa})}_{n\times m} = \mathrm{Tr}[(T^{(\mathrm{p})}_n)^m R_n] \,, \tag{4.31}$$
$$Z^{(\mathrm{ap})}_{n\times m} = \mathrm{Tr}(T^{(\mathrm{a})}_n)^m \,, \tag{4.32}$$
$$Z^{(\mathrm{aa})}_{n\times m} = \mathrm{Tr}[(T^{(\mathrm{a})}_n)^m R_n] \,. \tag{4.33}$$

The calculation of the numbers of rhombuses of each type is also straightforward. table 1 shows the number c of homogeneous coordinates for the model $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$ for several n and m.

The analyses is also applicable to the orbifold $\mathbb{C}^3/\mathbb{Z}_n$. The McKay quiver of the orbifold $\mathbb{C}^3/\mathbb{Z}_n$ is obtained from that of the orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ by the identification [10],

$$R_{i,j} \equiv R_{i-a,j+1} \,. \tag{4.34}$$

Thus if we consider configurations of spins which satisfies this identification, we are able to compute the toric generators and the degeneracy.

Acknowledgments

I would like to thank S. Hosono for valuable discussions.

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